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Families of abstract decision problems whose admissible sets intersect in a singleton

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Abstract
An abstract decision problem is an ordered pair where the first component is a nonempty and finite set of alternatives and the second component is an irreflexive relation on that set, called dominance relation. The admissible set of an abstract decision problem is the set of the maximal elements of the reflexive and transitive closure of the dominance relation. Given a finite sequence of abstract decision problems on the same set of alternatives, we give conditions on the dominance relations that guarantee that the intersection of all the admissible sets of the considered problems is a singleton as well as conditions that guarantee that the intersection is nonempty. We show then that such results allow to deduce some interesting facts about the resoluteness of the Schulze network solution and the Schulze social choice correspondence as well as some information about the existence of a (unique) common recurrent state for finite families of discrete-time homogeneous Markov chains.

1 Introduction

Many decision problems can be modelled by an abstract decision problem, namely a pair (X, R), where X is a nonempty and finite set of alternatives and R an irreflexive relation on X, called dominance relation. A major issue is to determine a sensible solution concept that allows to select one or more sets of alternatives for any given abstract decision problem. Several solution concepts have been proposed along the years: the stable set (Von Neumann and Morgenstern 1944), the generalized stable set (Van Deemen 1991), the socially stable set (Delver and Monsuur 2001), the m-stable set (Peris and Subiza 2013), the w-stable set (Han and Van Deemen 2016).

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The admissible set, also known as Schwartz set, GOCHA set and dynamic solution, is a further well-known solution concept for abstract decision problems (Kalai et al. 1976; Kalai and Schmeidler 1977; Schwartz 1972, 1986; Shenoy 1979, 1980). Given an abstract decision problem (X, R), the admissible set of (X, R) is the set of the maximal elements of the reflexive and transitive closure of R. Interpreting the dominance relation as a description of all the valid arguments that lead individuals of a society to shift from an alternative to another one in a social bargaining process, the alternatives in the admissible set correspond to the ones that, if considered at a certain stage of the bargaining process, cannot be definitely excluded by the individuals as final outcomes.

In this paper we focus on the admissible set and we show that it satisfies a remarkable and quite surprising property. Indeed, consider a nonempty and finite set of alternatives X. As the main result of the paper, we prove that if $(R_i)_{i=0}^{\alpha}$ is a sequence of irreflexive relations on X such that $R_0 = \varnothing$, $R_\alpha = \{(x, y) \in X^2 : x \neq y\}$ and, for every $i \in \{1, \ldots, \alpha\}$, $R_{i-1} \subseteq R_i$ and the size of $R_i \setminus R_{i-1}$ is 1, then the intersection of all the admissible sets of the abstract decision problems (X, R_i) is a singleton (Theorem 2). As an immediate consequence, we have that if $(R_i)_{i=0}^{\alpha}$ is a sequence of irreflexive relations on X such that, for every $i \in \{1, \ldots, \alpha\}$, $R_{i-1} \subseteq R_i$, then the intersection of all the admissible sets of the problems (X, R_i) is nonempty (Theorem 3). That last result has a very interesting interpretation. Assume that a society has to determine an alternative among the ones in a given set through a social bargaining process and that there are several informative scenarios, each of them providing different arguments to move across alternatives during the bargaining process. Assume further that those scenarios can be ordered in such a way that in any scenario individuals are allowed to apply all the arguments that are allowed in the previous scenarios. Then, we can conclude that there is at least an alternative having the property that, independently on the actual informative scenario available to individuals, if that alternative is taken into consideration at a certain stage of the bargaining process, for instance at the beginning of the discussion, it cannot be definitely excluded from the set of possible outcomes.

The aforementioned results, presented in Sect. 3, have applications in different contexts. The description of those applications is the content of Sects. 4, 5 and 6.

In Sect. 4 we propose some new properties of the so-called Schulze network solution, a special procedure that allows to select, for any given network, some of its vertices. That procedure, defined in Bubboloni and Gori (2018) on the basis of the voting method proposed in Schulze (2011), can be applied, for instance, for selecting the winners of a competition that can be represented by a network. We show that, for every network, the outcome of the Schulze network solution coincides with the intersection of the admissible sets of a suitable family of abstract decision problems (Theorem 4). On the basis of such a characterization result, we get then an alternative proof of the fact that the Schulze network solution is nonempty valued and we find out conditions on the capacities of the arcs of a network that guarantee that only one vertex is selected (Proposition 5). Moreover, using the obvious identification of an abstract decision problem with a network, we show that, for every abstract decision problem, the Schulze network solution

selects all and only the alternatives that are in the admissible set (Proposition 6). In other words, we show that the Schulze network solution can be seen as an extension of the admissible set to networks.

In Sect. 5, we focus on the Schulze social choice correspondence (SCC), a notable SCC that basically corresponds to the Schulze method (Schulze 2011) when individuals' preferences are linear orders on the set of alternatives, and we show that, for every preference profile, the outcome of the Schulze SCC coincides with the intersection of the admissible sets of a suitable family of abstract decision problems (Theorem 7). As a consequence of such a characterization result, we get an alternative proof of the fact that the Schulze SCC is nonempty valued and we provide conditions on preference profiles that are sufficient to make the Schulze SCC select exactly one alternative (Proposition 8).

Finally, in Sect. 6, we focus on a finite family of discrete-time homogeneous Markov chains over the same finite state space and we provide simple conditions on their transition matrices that guarantee that they have a unique common recurrent state as well as conditions that guarantee that they admit at least one common recurrent state.

The proof of the main theorem of the paper, namely Theorem 2, is given in Sect. 7. There, Theorem 2 is obtained as a consequence of a much more general result, namely Theorem 10. Such a result, whose statement and proof are quite technical, provides a characterization of the intersection of the admissible sets of the abstract decision problems (X, R_i), where $(R_i)_{i=0}^{\alpha}$ is a sequence of irreflexive relations on X such that, for every $i \in \{1, \dots, \alpha\}$, $R_{i-1} \subseteq R_i$ and the size of $R_i \setminus R_{i-1}$ is 1. Further generalizations of Theorems 2 and 3 are proposed in Section 8.

2 Preliminary notation

We assume $0 \notin \mathbb{N}$ and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let X be a finite set. We set $X_d^2 = \{(x, y) \in X^2 : x = y\}$ and $X_*^2 = \{(x, y) \in X^2 : x \neq y\}$. The size of X is denoted by $|X|$. A relation on X is a subset of X^2. Given a relation R on X, we say that R is reflexive if, for every $x \in X$, $(x, x) \in R$; irreflexive if, for every $x \in X$, $(x, x) \notin R$; asymmetric if, for every $x, y \in X$, $(x, y) \in R$ implies $(y, x) \notin R$; transitive if, for every $x, y, z \in X$, $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$; antisymmetric if, for every $x, y \in X$, $(x, y) \in R$ and $(y, x) \in R$ imply $x = y$; complete if, for every $x, y \in X$, $(x, y) \in R$ or $(y, x) \in R$; a linear order if R is complete, transitive and antisymmetric.

3 The admissible set

An abstract decision problem is an ordered pair (X, R), where X is a nonempty and finite set called set of alternatives and R is an irreflexive relation on X called dominance relation. If $(x, y) \in R$, we say that x dominates y (x is a dominator of y).

Given an abstract decision problem (X, R), the elements of X are interpreted as mutually exclusive alternatives from which a society, or a group of individuals, have to make a choice through a social bargaining process. The relation R describes the valid arguments that allow to move from an alternative to another one. More precisely, assume that $y \in X$ is taken into consideration by the individuals at a certain stage of the bargaining process. If $\{x \in X : (x, y) \in R\} \neq \emptyset$, then the alternatives in the set $\{x \in X : (x, y) \in R\}$ are interpreted as the alternatives that will be considered by the individuals at the next stage. In other words, in the bargaining process individuals are assumed to move from y to one of the alternatives that dominate y, if any. If instead y has no dominator, that is $\{x \in X : (x, y) \in R\} = \emptyset$, then the bargaining process has no reason to go on and y is indeed the final outcome.

The theory of abstract decision problems investigates which are the alternatives that can be considered potential outcomes of a given problem (X, R). The admissible set is a well-known solution concept introduced by Kalai et al. (1976) and Kalai and Schmeidler (1977). Equivalent definitions of that concept are given by Schwartz (1972, 1986) and Shenoy (1979, 1980).[1] Let us present the definition of such a set.

Let (X, R) be an abstract decision problem. Let us denote by R^τ the reflexive and transitive closure of R, that is, the smallest reflexive and transitive relation on X containing R. Note that $R = \emptyset$ implies $R^\tau = X_d^2$ and $R = X_*^2$ implies $R^\tau = X^2$. A path in R is a sequence $(x_j)_{j=1}^m$, where $m \geq 2$, $x_1, \ldots, x_m$ are distinct elements of X and, for every $j \in \{1, \ldots, m-1\}$, $(x_j, x_{j+1}) \in R$. If $x, y \in X$ are distinct, a path from x to y in R is a path $(x_j)_{j=1}^m$ in R such that $x_1 = x$ and $x_m = y$. It is easily seen that

$$R^\tau = X_d^2 \cup \left\{(x, y) \in X_*^2 : \text{ there exists a path from } x \text{ to } y \text{ in } R\right\}.$$

The relation R^τ has a natural interpretation. Indeed, consider $x, y \in X$. Then $(x, y) \in R^\tau$ means that if y is the alternative taken into consideration by the individuals at a certain stage of the bargaining process, then x is the alternative selected at that stage (that is, $x = y$) or x can be selected at some future stage. Analogously, $(x, y) \notin R^\tau$ means that if y is the alternative taken into consideration by the individuals at a certain stage of the bargaining process, then the individuals are not considering the alternative x at that stage (that is, $x \neq y$) and they will never consider x in the future.

Definition 1 Let (X, R) be an abstract decision problem. The admissible set of (X, R) is the set

$$A(X, R) = \{x \in X : \forall y \in X,\ (y, x) \in R^\tau \text{ implies } (x, y) \in R^\tau\}.$$

The elements of the admissible set are then the alternatives having the property that they cannot be definitely excluded as final outcomes of the bargaining process once they are taken into consideration at a certain stage. More precisely, an alternative x is in the admissible set if the fact of being considered by the individuals at a

[1] See also Van Deemen (1997).

certain stage of the bargaining process along with the fact that another alternative is considered in one of the next stages does not exclude the individuals to consider x again in the future.[2]

Let us consider now an abstract decision problem (X, R). It is well-known that $A(X,R) \neq \emptyset$.[3] Moreover, it is immediate to show that if $R = \emptyset$ or $R = X_*^2$, then $A(X,R) = X$. In particular, if $|X| = 1$, then $A(X,R) = X$.

3.1 Main result

Before stating our main result, let us propose an example. Consider $X = \{1,2,3\}$ and let

$$a_1 = (1,2),\ a_2 = (2,1),\ a_3 = (2,3),\ a_4 = (3,1),\ a_5 = (3,2),\ a_6 = (1,3).$$

Consider then the sequence $(R_i)_{i=0}^6$ of irreflexive relations on X such that $R_0 = \emptyset$ and, for every $i \in \{1,\dots,6\}$, $R_i = \{a_1,\dots,a_i\}$. Thus, for every $i \in \{1,\dots,6\}$, R_i is obtained by adding a new element to R_{i-1}. Note also that $R_6 = X_*^2$. An easy computation shows that

$$A(X,R_0) = \{1,2,3\},\ A(X,R_1) = \{1,3\},\quad A(X,R_2) = \{1,2,3\},\ A(X,R_3) = \{1,2\},$$
$$A(X,R_4) = \{1,2,3\},\ A(X,R_5) = \{1,2,3\},\ A(X,R_6) = \{1,2,3\}.$$

From the example it is possible to understand that the way the admissible set changes when an element is added to a relation does not seem to follow any simple rule. In particular, $R \subseteq \hat{R}$ does not imply, in general, either $A(X,R) \subseteq A(X,\hat{R})$ (as R_0 and R_1 show) or $A(X,\hat{R}) \subseteq A(X,R)$ (as R_1 and R_2 show). In the considered example, however, $R \subseteq \hat{R}$ always implies $A(X,R) \cap A(X,\hat{R}) \neq \emptyset$. More precisely, all the considered admissible sets have a unique common element, namely alternative 1. Quite surprisingly, that fact is a general result as shown by Theorem 2, the main result of the paper. The proof of Theorem 2 is proposed in Sect. 7. In that section Theorem 2 is shown to be a consequence of a more general but rather technical result, namely Theorem 10.

Theorem 2 *Let X be a nonempty and finite set, $\alpha \in \mathbb{N}$ and $(R_i)_{i=0}^{\alpha}$ be a sequence of irreflexive relations on X. Assume that $R_0 = \emptyset$, $R_\alpha = X_*^2$ and, for every $i \in \{1,\dots,\alpha\}$, $R_{i-1} \subseteq R_i$ and $|R_i \setminus R_{i-1}| = 1$. Then, there exists $x^* \in X$, such that $\bigcap_{i=0}^{\alpha} A(X,R_i) = \{x^*\}$.*

Note that the assumptions on $(R_i)_{i=0}^{\alpha}$ in Theorem 2 also imply $|X| \geq 2$ and $\alpha = |X_*^2|$. As the example previously discussed shows, the intersection of the admissible sets is a singleton even though none of the admissible sets considered in the

[2] Note that when (X, R) is a tournament, that is, R is asymmetric and, for every $x, y \in X$ with $x \neq y$, we have that $(x,y) \in R$ or $(y,x) \in R$, then $A(X,R)$ coincides with the top cycle associated with (X,R).

[3] See, for instance, Theorem 4.1 in Shenoy (1980).

intersection is a singleton. The next result is a simple but remarkable consequence of Theorem 2.

Theorem 3 *Let X be a nonempty and finite set, $\alpha \in \mathbb{N}$ and $(R_i)_{i=0}^{\alpha}$ be a sequence of irreflexive relations on X. Assume that, for every $i \in \{1, \dots, \alpha\}$, $R_{i-1} \subseteq R_i$. Then, $\bigcap_{i=0}^{\alpha} A(X, R_i) \neq \emptyset$.*

Proof Assume first that $|X| = 1$. Thus, for every $i \in \{0, \dots, \alpha\}$, $R_i = \emptyset$ and $A(X, R_i) = X$. That immediately implies $\bigcap_{i=0}^{\alpha} A(X, R_i) = X \neq \emptyset$. Assume next that $|X| \geq 2$. It is simple to show that there exists a sequence $(S_j)_{j=0}^{\beta}$ of irreflexive relations on X having the following properties: $\beta = |X_*^2|$; $S_0 = \emptyset$; $S_\beta = X_*^2$; for every $j \in \{1, \dots, \beta\}$, $S_{j-1} \subseteq S_j$ and $|S_j \setminus S_{j-1}| = 1$; for every $i \in \{0, \dots, \alpha\}$, there exists $j \in \{0, \dots, \beta\}$ such that $S_j = R_i$. Thus, the sequence $(S_j)_{j=0}^{\beta}$ satisfies the assumptions of Theorem 2 and then there exists $x^* \in X$ such that $\bigcap_{j=0}^{\beta} A(X, S_j) = \{x^*\}$. Since $\bigcap_{j=0}^{\beta} A(X, S_j) \subseteq \bigcap_{i=0}^{\alpha} A(X, R_i)$, we conclude that $x^* \in \bigcap_{i=0}^{\alpha} A(X, R_i) \neq \emptyset$, as desired. □

4 The Schulze network solution

Let X be a finite set with $|X| \geq 2$. A network on X is a triple $N = (X, X_*^2, c)$, where c is a function from X_*^2 to $\mathbb{R}$. We say that X is the set of vertices of N, X_*^2 is the set of arcs of N and c is the capacity associated with N. The set of networks on X is denoted by $\mathcal{N}$. A network solution is a correspondence from $\mathcal{N}$ to X, that is, a procedure for selecting some of the vertices of any given network on X. Networks can be used to mathematically represent a variety of social situations. As a remarkable example, consider a competition where a set X of teams played a certain number of matches among each other. Such a competition can be naturally represented by the network $N = (X, X_*^2, c)$, where, for every $(x, y) \in X_*^2$, $c(x, y)$ is the number of matches in which x beat y. When networks are interpreted as representations of competitions, network solutions represent procedures to determine the set of winners. Network solutions (as well as procedures for ranking the vertices of a network) are extensively studied in the literature (Laslier 1997; Langville and Meyer 2012; González-Díaz et al. 2014).

On the basis of the Schulze method introduced by Schulze (2011), Bubboloni and Gori (2018) propose a special network solution called Schulze network solution. Let us recall its definition. Let $N = (X, X_*^2, c) \in \mathcal{N}$. A path in N is a sequence $(x_j)_{j=1}^{m}$, where $m \geq 2$, $x_1, \dots, x_m$ are distinct elements of X. Consider now $x, y \in X$ distinct. A path from x to y in N is a path $(x_j)_{j=1}^{m}$ in N such that $x_1 = x$ and $x_m = y$. The set of paths from x to y in N is denoted by $\Gamma(N, x, y)$. Note that $\Gamma(N, x, y)$ is nonempty and finite. Given $\gamma = (x_j)_{j=1}^{m} \in \Gamma(N, x, y)$, let

$$\delta^N(\gamma) = \min\{c(x_j, x_{j+1}) : j \in \{1, ...m-1\}\}.$$

Define then

$$s^N_{xy} = \max\{\delta^N(\gamma) : \gamma \in \Gamma(N, x, y)\}.$$

The Schulze network solution, denoted by $\mathbf{S}$, is defined, for every $N \in \mathcal{N}$, by

$$\mathbf{S}(N) = \{x \in X : \forall y \in X \setminus \{x\}\ s^N_{xy} \geq s^N_{yx}\}.$$

Let $N = (X, X^2_*, c) \in \mathcal{N}$. For every $\mu \in \mathbb{R}$, consider the irreflexive relation on X given by

$$\Sigma_\mu(N) = \{(x, y) \in X^2_* : c(x, y) \geq \mu\}. \tag{1}$$

Note that, for every $\mu, \mu' \in \mathbb{R}$ with $\mu \leq \mu'$, we have that $\Sigma_{\mu'}(N) \subseteq \Sigma_\mu(N)$.

The next result provides a characterization of the Schulze network solution.

Theorem 4 *Let $N = (X, X^2_*, c) \in \mathcal{N}$ and $K \subseteq \mathbb{R}$ with $\mathrm{Im}(c) \subseteq K$. Then*

$$\mathbf{S}(N) = \bigcap_{\mu \in K} A(X, \Sigma_\mu(N)).$$

Proof Define $\mathbf{M}(N) = \bigcap_{\mu \in K} A(X, \Sigma_\mu(N))$ and prove that $\mathbf{S}(N) = \mathbf{M}(N)$. For every $\mu \in K$, let

$$Q_\mu(N) = \left\{(x, y) \in X^2_* : s^N_{xy} \geq \mu\right\} \cup X^2_d. \tag{2}$$

We claim that, for every $\mu \in K$,

$$Q_\mu(N) = \Sigma_\mu(N)^\tau. \tag{3}$$

Let us prove first that $Q_\mu(N) \subseteq \Sigma_\mu(N)^\tau$. Let $(x, y) \in Q_\mu(N)$. If $x = y$, we have that $(x, y) \in \Sigma_\mu(N)^\tau$. Assume then that $x \neq y$. Thus, $s^N_{xy} \geq \mu$ so that there exists $\gamma = (x_j)_{j=1}^m \in \Gamma(N, x, y)$ such that $\delta^N(\gamma) \geq \mu$. Then, for every $j \in \{1, \ldots, m-1\}$, $c(x_j, x_{j+1}) \geq \mu$. As a consequence, we deduce that γ is in fact a path from x to y in $\Sigma_\mu(N)$, so that $(x, y) \in \Sigma_\mu(N)^\tau$. We conclude then that $Q_\mu(N) \subseteq \Sigma_\mu(N)^\tau$. Let us prove next that $\Sigma_\mu(N)^\tau \subseteq Q_\mu(N)$. Let $(x, y) \in \Sigma_\mu(N)^\tau$. If $x = y$, we have that $(x, y) \in Q_\mu(N)$. Assume then that $x \neq y$. Thus, there exists a path $\gamma = (x_j)_{j=1}^m$ from x to y in $\Sigma_\mu(N)$. We have then that, for every $j \in \{1, \ldots, m-1\}$, $c(x_j, x_{j+1}) \geq \mu$. As a consequence, $\delta^N(\gamma) \geq \mu$ so that $s^N_{xy} \geq \mu$. That implies $(x, y) \in Q_\mu(N)$. We conclude then that $\Sigma_\mu(N)^\tau \subseteq Q_\mu(N)$. That completes the proof of (3).

We prove now that $\mathbf{S}(N) = \mathbf{M}(N)$ by showing that, for every $x \in X$, $x \notin \mathbf{S}(N)$ if and only if $x \notin \mathbf{M}(N)$. Consider then $x \in X$ and the following facts:

(a_1) $x \notin \mathbf{S}(N)$;
(a_2) there exists $y \in X \setminus \{x\}$ such that $s^N_{yx} > s^N_{xy}$;
(a_3) there exist $\mu \in K$ and $y \in X \setminus \{x\}$ such that $s^N_{yx} \geq \mu$ and $s^N_{xy} < \mu$;
(a_4) there exist $\mu \in K$ and $y \in X \setminus \{x\}$ such that $(y,x) \in Q_\mu(N)$ and $(x,y) \notin Q_\mu(N)$;
(a_5) there exist $\mu \in K$ and $y \in X \setminus \{x\}$ such that $(y,x) \in \Sigma_\mu(N)^r$ and $(x,y) \notin \Sigma_\mu(N)^r$;
(a_6) there exists $\mu \in K$ such that $x \notin A(X, \Sigma_\mu(N))$;
(a_7) $x \notin \mathbf{M}(N)$.

By (2), (3), the definition of admissible set and the fact that, for every $x, y \in X$ distinct, $s^N_{xy} \in \mathrm{Im}(c)$, it is immediate to prove that, for every $i \in \{1, \dots, 6\}$, (a_i) is equivalent to (a_{i+1}). Thus, (a_1) is equivalent to (a_7) and the proof is completed. □

Let $N = (X, X^2_*, c) \in \mathcal{N}$ with $\mathrm{Im}(c) \subseteq \mathbb{N}_0$. Theorem 4 provides a new way to interpret the vertices selected by $\mathbf{S}$ when N is interpreted as a model for a competition. Indeed, consider $x, y \in X$ with $x \neq y$ and $\mu \in \mathbb{N}_0$ and assume that there is a path $(x_j)_{j=1}^m$ from x to y in $\Sigma_\mu(N)$. Thus, for every $j \in \{1, \dots, m-1\}$, we have that $c(x_j, x_{j+1}) \geq \mu$, that is, x_j beat x_{j+1} at least μ times. Thus, we can say that x directly (if $m = 2$) or indirectly (if $m \geq 3$) beat y at least μ times. By Theorem 4 we can conclude then that a team x is selected by $\mathbf{S}$ if and only if, for every $\mu \in \mathbb{N}_0$, the fact that x is directly or indirectly beaten by another team y at least μ times implies that in turn x directly or indirectly beat y at least μ times.

It is known that $\mathbf{S}$ is nonempty valued.[4] Thanks to Theorems 2, 3 and 4, we provide an alternative proof of that fact as well as a simple sufficient condition for resoluteness of $\mathbf{S}$.

Proposition 5 *Let $N = (X, X^2_*, c) \in \mathcal{N}$. Then $\mathbf{S}(N) \neq \emptyset$. Moreover, if c is injective, then $|\mathbf{S}(N)| = 1$.*

Proof Since X is finite with $|X| \geq 2$, we have that X^2_* is nonempty and finite so that $\mathrm{Im}(c)$ is nonempty and finite, as well. Let $|\mathrm{Im}(c)| = \alpha \in \mathbb{N}$ and consider $\mu_1, \dots, \mu_\alpha \in \mathbb{R}$ such that $\mathrm{Im}(c) = \{\mu_1, \dots, \mu_\alpha\}$ and such that, for every $i, j \in \{1, \dots, \alpha\}$ with $i < j$, $\mu_i > \mu_j$. Let $\mu_0 \in \mathbb{R}$ be such that $\mu_0 > \mu_1$ and set $K = \mathrm{Im}(c) \cup \{\mu_0\}$. Moreover, for every $i \in \{0, \dots, \alpha\}$, set $R_i = \Sigma_{\mu_i}(N)$. Thus, by Theorem 4, we have that

$$\mathbf{S}(N) = \bigcap_{\mu \in K} A(X, \Sigma_\mu(N)) = \bigcap_{i=0}^{\alpha} A(X, R_i). \tag{4}$$

[4] That fact is, for instance, a consequence of Propositions 2 and 54 in Bubboloni and Gori (2018). Note that Proposition 54 is actually about networks having capacity whose values are nonnegative integers but it can be easily generalized to the case of networks with real-valued capacity.

Observe now that $R_0 = \emptyset$, $R_\alpha = X_*^2$ and, for every $i \in \{1, \dots, \alpha\}$, $R_{i-1} \subseteq R_i$. Thus, in particular, the sequence $(R_i)_{i=0}^{\alpha}$ satisfies the assumptions of Theorem 3. We conclude then that $\mathbf{S}(N) \neq \emptyset$.

Assume now that c is injective. We claim that, for every $i \in \{1, \dots, \alpha\}$, $|R_i \setminus R_{i-1}| = 1$. Consider $i \in \{1, \dots, \alpha\}$ and denote by (x_i, y_i) the unique element of X_*^2 such that $c(x_i, y_i) = \mu_i$. Thus, $(x_i, y_i) \in R_i$ and $(x_i, y_i) \notin R_{i-1}$ so that $|R_i \setminus R_{i-1}| \geq 1$. Assume now by contradiction that $|R_i \setminus R_{i-1}| \geq 2$. Thus, there exist $(x_1, y_1), (x_2, y_2) \in X_*^2$ with $(x_1, y_1) \neq (x_2, y_2)$ that belong to R_i and do not belong to R_{i-1}. Thus, for every $k \in \{1, 2\}$, $c(x_k, y_k) \geq \mu_i$ and $c(x_k, y_k) < \mu_{i-1}$, that is, $c(x_k, y_k) = \mu_i$. That contradicts the fact that c is injective. We conclude then that $|R_i \setminus R_{i-1}| = 1$. Thus, the sequence $(R_i)_{i=0}^{\alpha}$ satisfies the assumptions of Theorem 2. By (4), we conclude then that there exists $x^* \in X$ such that $\mathbf{S}(N) = \{x^*\}$. □

Observe now that, given an irreflexive relation R on X, R can be naturally identified with the network $N_R = (X, X_*^2, c_R)$, where c_R is defined, for every $(x, y) \in X_*^2$, by

$$c_R(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R, \\ 0 & \text{if } (x, y) \notin R. \end{cases}$$

The next proposition shows that the Schulze network solution can be seen as a generalization of the concept of admissible set to networks.

Proposition 6 *Let R be an irreflexive relation on X. Then $\mathbf{S}(N_R) = A(X, R)$.*

Proof Since $\mathrm{Im}(c_R) \subseteq \{0, 1\}$, by Theorem 4, we have that $\mathbf{S}(N_R) = \bigcap_{\mu \in \{0,1\}} A(X, \Sigma_\mu(N_R))$. Moreover, $\Sigma_0(N_R) = X_*^2$ and $\Sigma_1(N_R) = R$ so that $A(X, \Sigma_0(N_R)) = X$ and $A(X, \Sigma_1(N_R)) = A(X, R)$. Thus, we conclude that $\mathbf{S}(N_R) = A(X, R)$. □

5 The Schulze social choice correspondence

Let X and I be finite sets with $|X| \geq 2$ and $|I| \geq 2$. We interpret X as the set of alternatives and I as the set of individuals. We denote by $\mathcal{L}(X)$ the set of linear orders on X. Given $q \in \mathcal{L}(X)$ and $x, y \in X$, we write $x \succ_q y$ when $(x, y) \in q$ and $(y, x) \notin q$. The elements of $\mathcal{L}(X)$ are interpreted as individual preferences. The set of functions from I to $\mathcal{L}(X)$ is denoted by $\mathcal{L}(X)^I$. Each element p in $\mathcal{L}(X)^I$ is called preference profile and represents the complete description of individual preferences: for every $i \in I$, $p(i)$ is interpreted as the preferences of individual i. Given $p \in \mathcal{L}(X)^I$, we denote by c_p the function from X_*^2 to $\mathbb{N}_0$ defined, for every $(x, y) \in X_*^2$, by $c_p(x, y) = |\{i \in I : x \succ_{p(i)} y\}|$. Thus, $c_p(x, y)$ counts the number of individuals preferring x to y according to the preferences described by p. Given $p \in \mathcal{L}(X)^I$, we denote by N_p the network (X, X_*^2, c_p).

A social choice correspondence (scc) is a correspondence from $\mathcal{L}(X)^I$ to X. The Schulze scc, here denoted by S, is defined, for every $p \in \mathcal{L}(X)^I$, by $S(p) = \mathbf{S}(N_p)$,

where **S** is the Schulze network solution defined in the previous section. The Schulze SCC is an important SCC that coincides with the Schulze method when individual preferences are linear orders and, using the words of Schulze (2011), the strength of the link xy is measured by margin for all $x, y \in X$ with $x \neq y$. It is known that S is nonempty valued and satisfies many desirable properties (Schulze 2011, Schulze 2020). On the basis of the theory developed, we propose here a characterization of S based on the relations of the type $\Sigma_\mu(N)$ defined in (1). In what follows, let $\mu^s = \min\left(\mathbb{N} \cap \left(\frac{|I|}{2}, |I|\right]\right)$.

Theorem 7 *Let $p \in \mathcal{L}(X)^I$. Then*

$$S(p) = \bigcap_{\mu \in \mathbb{N} \cap \left(\frac{|I|}{2}, |I|\right]} A(X, \Sigma_\mu(N_p)).$$

Proof Since $\mathrm{Im}(c) \subseteq \{0, \ldots, |I|\}$, applying Theorem 4, we have that

$$S(p) = \mathbf{S}(N_p) = \bigcap_{\mu=0}^{|I|} A(X, \Sigma_\mu(N_p)).$$

In order to complete the proof it is sufficient to show that, for every $\mu \in \mathbb{N}_0$ with $\mu < \mu^s$, $A(X, \Sigma_{\mu^s}(N_p)) \subseteq A(X, \Sigma_\mu(N_p))$. Consider then $\mu \in \mathbb{N}_0$ with $\mu < \mu^s$ and let $x \in A(X, \Sigma_{\mu^s}(N_p))$. We have to prove that $x \in A(X, \Sigma_\mu(N_p))$. Consider then $y \in X$, assume that $(y, x) \in \Sigma_\mu(N_p)^\tau$ and prove that $(x, y) \in \Sigma_\mu(N_p)^\tau$. First of all, note that $\mu < \mu^s$ implies that $\Sigma_{\mu^s}(N_p) \subseteq \Sigma_\mu(N_p)$ and then $\Sigma_{\mu^s}(N_p)^\tau \subseteq \Sigma_\mu(N_p)^\tau$. If $(x, y) \in \Sigma_{\mu^s}(N_p)$, then $(x, y) \in \Sigma_\mu(N_p)$ so that $(x, y) \in \Sigma_\mu(N_p)^\tau$, as desired. If $(y, x) \in \Sigma_{\mu^s}(N_p)$, then $(y, x) \in \Sigma_{\mu^s}(N_p)^\tau$. Since $x \in A(X, \Sigma_{\mu^s}(N_p))$, we have that $(x, y) \in \Sigma_{\mu^s}(N_p)^\tau$ so that $(x, y) \in \Sigma_\mu(N_p)^\tau$, as desired. Assume finally that $(x, y) \notin \Sigma_{\mu^s}(N_p)$ and $(y, x) \notin \Sigma_{\mu^s}(N_p)$. If $|I|$ is odd, $c_p(x, y) \leq \frac{|I|-1}{2}$ and $c_p(y, x) \leq \frac{|I|-1}{2}$ so that $|I| = c_p(x, y) + c_p(y, x) \leq |I| - 1$, a contradiction. Thus, $|I|$ must be even and $c_p(x, y) \leq \frac{|I|}{2}$ and $c_p(y, x) \leq \frac{|I|}{2}$. Since $|I| = c_p(x, y) + c_p(y, x)$, we necessarily have $c_p(x, y) = c_p(y, x) = \frac{|I|}{2}$. Since $\mu < \mu^s = \frac{|I|}{2} + 1$, we have that $\mu \leq \frac{|I|}{2}$. Thus, in particular, $(x, y) \in \Sigma_\mu(N_p)$ so that $(x, y) \in \Sigma_\mu(N_p)^\tau$, as desired. □

Theorem 7 allows to get a new and appealing interpretation of the alternatives selected by S. Indeed, assume that individuals in the society have preferences described by $p \in \mathcal{L}(X)^I$. Then, an alternative x belongs to $S(p)$ if and only if, for every possible majority threshold $\mu \in \mathbb{N} \cap \left(\frac{|I|}{2}, |I|\right]$, x cannot be definitely excluded by the individuals as final outcome of a social bargaining process based on the qualified majority principle related to μ, provided that x is taken into consideration in the bargaining process at a certain stage. Note also that Theorem 7 immediately implies the already known fact that S satisfies the Schwartz criterion, namely that, for every $p \in \mathcal{L}(X)^I$, $S(p) \subseteq A(X, \Sigma_{\mu^s}(N_p))$ (Schulze 2020).

The next proposition provides an alternative proof of the fact, already proved in Schulze (2011), that S is nonempty valued. Moreover, a sufficient condition for resoluteness of S is given.

Proposition 8 *Let $p \in \mathcal{L}(X)^I$. Then $S(p) \neq \emptyset$. Moreover, if c_p is injective, then $|S(p)| = 1$.*

Proof Simply use the definition of S and Proposition 5. □

6 Markov chains

Theorems 2 and 3 immediately allow to establish conditions that make a finite family of discrete-time homogeneous Markov chains over the same finite state space admit a (unique) common recurrent state.

Consider a probability space $(\Omega, \mathcal{F}, P)$, a nonempty and finite state space X and $\alpha \in \mathbb{N}$. For every $i \in \{0, \dots, \alpha\}$ and $n \in \mathbb{N}_0$, let V_n^i be a random variable defined on $(\Omega, \mathcal{F}, P)$ with values in X. Assume then that, for every $i \in \{0, \dots, \alpha\}$, the sequence $\mathcal{M}^i = (V_n^i)_{n \in \mathbb{N}_0}$ is a homogeneous Markov chain with transition probability matrix $(p_{xy}^i)_{x,y \in X}$ and set of recurrent states $RS(\mathcal{M}^i)$. It is known that, for every $i \in \{0, \dots, \alpha\}$, $RS(\mathcal{M}^i) = A(X, R_i)$, where $R_i = \{(x, y) \in X_*^2 : p_{yx}^i > 0\}$.

As a consequence, if we know that $R_0 = \emptyset$, $R_\alpha = X_*^2$ and, for every $i \in \{1, \dots, \alpha\}$, $R_{i-1} \subseteq R_i$ and $|R_i \setminus R_{i-1}| = 1$, then we can apply Theorem 2 and deduce that there exists $x^* \in X$ such that $\bigcap_{i=0}^{\alpha} RS(\mathcal{M}^i) = \{x^*\}$. If instead we only know that, for every $i \in \{1, \dots, \alpha\}$, $R_{i-1} \subseteq R_i$, then we can apply Theorem 3 and deduce that $\bigcap_{i=0}^{\alpha} RS(\mathcal{M}^i) \neq \emptyset$. Thus, if the sequence $(\mathcal{M}^i)_{i=0}^{\alpha}$ has the property that, for every $i \in \{1, \dots, \alpha\}$, all the transitions among different states that are available in $\mathcal{M}^{i-1}$ are available in $\mathcal{M}^i$ too, then there exists at least a state $x^* \in X$ that is recurrent for all the Markov chains in the sequence. Thus, if x^* is assumed to be the starting state, each Markov chain in the sequence $(\mathcal{M}^i)_{i=0}^{\alpha}$ will return with probability 1 to state x^* within a finite time.

7 A generalization of Theorem 2

This section is devoted to state and prove a general and technical result, namely Theorem 10, that implies Theorem 2 as a corollary. In order to state Theorem 10, it is necessary to introduce some preliminary definitions and results.

Let (X, R) be an abstract decision problem. Let E_R be the equivalence relation on X defined as

$$E_R = \left\{(x, y) \in X^2 : (x, y) \in R^\tau \text{ and } (y, x) \in R^\tau\right\}.$$

We denote by $\mathscr{S}(X, R)$ the quotient set of X by E_R. Recall that $\mathscr{S}(X, R)$ is a partition of X and that, given $Y \subseteq X$, we have that $Y \in \mathscr{S}(X, R)$ if and only if $Y \neq \emptyset$ and there exists $y^* \in Y$ such that

- for every $y \in Y, (y, y^*) \in E_R$,
- for every $x \in X, (x, y^*) \in E_R$ implies $x \in Y$.

The elements of $\mathscr{S}(X, R)$ are called strong components of (X, R). Given $x \in X$, we denote by $S(X, R; x)$ the unique element of $\mathscr{S}(X, R)$ having x as element. Note that, given $Y \in \mathscr{S}(X, R)$ and $x \in X$, we have that there exists $y \in Y$ such that $(x, y) \in R^\tau$ if and only if, for every $y \in Y, (x, y) \in R^\tau$. We set

$$\mathscr{A}(X, R) = \{Y \in \mathscr{S}(X, R) : \forall x \in X \setminus Y \; \forall y \in Y, (x, y) \notin R^\tau\}.$$

Thus, given $Y \in \mathscr{A}(X, R)$, $x \in X$ and $y \in Y$, if $(x, y) \in R^\tau$ then $x \in Y$ so that $(y, x) \in R^\tau$.

Theorem 9 below states, within our notation, a well-known fact.[5] For the sake of completeness, we give the proof.

Theorem 9 *Let (X, R) be an abstract decision problem. Then $\mathscr{A}(X, R) \neq \emptyset$ and $A(X, R) = \bigcup_{Y \in \mathscr{A}(X,R)} Y$. In particular, $A(X, R) \neq \emptyset$.*

Proof Let us prove first that $\mathscr{A}(X, R) \neq \emptyset$. Consider the relation $\mathfrak{R}$ on $\mathscr{S}(X, R)$ defined as

$$\mathfrak{R} = \left\{(Y, Y') \in \mathscr{S}(X, R)^2 : Y \neq Y' \text{ and } \exists x \in Y \, \exists y \in Y' \text{ such that } (x, y) \in R^\tau\right\}.$$

Clearly $\mathfrak{R}$ is irreflexive. Moreover, we have that $\mathfrak{R}$ is acyclic. Indeed, assume by contradiction that $\mathfrak{R}$ admits a cycle, that is, there exists a sequence $(Y_j)_{j=1}^m$, where $m \geq 3$, $Y_1, \ldots, Y_{m-1}$ are distinct elements of $\mathscr{S}(X, R)$, $Y_m = Y_1$ and, for every $j \in \{1, \ldots, m-1\}$, $(Y_j, Y_{j+1}) \in \mathfrak{R}$. Thus, for every $j \in \{1, \ldots, m-1\}$, there exist $x_j \in Y_j$ and $y_j \in Y_{j+1}$ such that $(x_j, y_j) \in R^\tau$. Since, for every $j \in \{1, \ldots, m-2\}$, we have that $x_{j+1} \in Y_{j+1}$ and $y_j \in Y_{j+1}$, we deduce that $(y_j, x_{j+1}) \in R^\tau$. By transitivity of R^τ, we have then that $(x_1, x_{m-1}) \in R^\tau$. Moreover, $x_1 \in Y_1$ and $y_{m-1} \in Y_m = Y_1$ so that $(y_{m-1}, x_1) \in R^\tau$. By $(x_{m-1}, y_{m-1}) \in R^\tau$ and transitivity of R^τ, we also get $(x_{m-1}, x_1) \in R^\tau$. We deduce then that $(x_1, x_{m-1}) \in E_R$ that implies the contradiction $Y_1 = Y_{m-1}$.

Since $\mathfrak{R}$ is irreflexive and acyclic, there exists $Y^* \in \mathscr{S}(X, R)$ such that, for every $Y \in \mathscr{S}(X, R)$, $(Y, Y^*) \notin \mathfrak{R}$.[6] Consider now $x \in X \setminus Y^*$ and $y \in Y^*$. Since $x \in S(X, R, x) \neq Y^*$ and $(S(X, R, x), Y^*) \notin \mathfrak{R}$, we have that $(x, y) \notin R^\tau$. That implies that $Y^* \in \mathscr{A}(X, R)$ so that $\mathscr{A}(X, R) \neq \emptyset$.

Let us prove now the equality $A(X, R) = \bigcup_{Y \in \mathscr{A}(X,R)} Y$. Consider $y \in A(X, R)$. We have that $S(X, R; y) \in \mathscr{A}(X, R)$. Indeed, assume by contradiction that there exists $x \in X \setminus S(X, R; y)$ such that $(x, y) \in R^\tau$. Then, since $y \in A(X, R)$, we also have that $(y, x) \in R^\tau$. Thus, $(x, y) \in E_R$ so that $x \in S(X, R, y)$, a contradiction. Since $y \in S(X, R; y)$, we have that $y \in \bigcup_{Y \in \mathscr{A}(X,R)} Y$. Thus, $A(X, R) \subseteq \bigcup_{Y \in \mathscr{A}(X,R)} Y$. Assume now that $y \in \bigcup_{Y \in \mathscr{A}(X,R)} Y$. Thus, there exists $Y \in \mathscr{A}(X, R)$ such that $y \in Y$. Consider

[5] See, for instance, Theorem 5 in Kalai and Schmeidler (1977).

[6] See Proposition 1.4.2 in Bang-Jensen and Gutin (2008).

now $x \in X$ such that $(x, y) \in R^\tau$. Then $x \in Y$ so that $(y, x) \in R^\tau$. That proves that $y \in A(X, R)$. We deduce then the inclusion $\bigcup_{Y \in \mathscr{A}(X,R)} Y \subseteq A(X, R)$. □

We are now ready to state the main result of this section, that is, Theorem 10. That theorem establishes the equality between the intersection of the admissible sets of the abstract decision problems (X, R_i), where $(R_i)_{i=0}^\alpha$ is a sequence of irreflexive relations such that, for every $i \in \{1, \dots, \alpha\}$, $R_{i-1} \subseteq R_i$ and $|R_i \setminus R_{i-1}| = 1$, and the union of certain special elements of $\mathscr{A}(X, R_0)$.

Theorem 10 *Let X be a nonempty and finite set, $\alpha \in \mathbb{N}$, $(R_i)_{i=0}^\alpha$ be a sequence of irreflexive relations on X and $((\hat{y}_i, \hat{z}_i))_{i=1}^\alpha$ be a sequence in X_*^2. Assume that, for every $i \in \{1, \dots, \alpha\}$, $R_{i-1} \subseteq R_i$ and $R_i \setminus R_{i-1} = \{(\hat{y}_i, \hat{z}_i)\}$. For every $i \in \{1, \dots, \alpha\}$, say that $\hat{z}_i$ satisfies the property $\mathfrak{P}_i$ if one of the following facts holds true*:

- $\hat{z}_i \in A(X, R_{i-1})$ and $(\hat{z}_i, \hat{y}_i) \notin R_{i-1}^\tau$;
- $\hat{z}_i \in A(X, R_{i-1})$, $(\hat{z}_i, \hat{y}_i) \in R_{i-1}^\tau$ and there exists $\hat{x}_i \in A(X, R_{i-1}) \setminus S(X, R_{i-1}; \hat{z}_i)$ such that $(\hat{x}_i, \hat{y}_i) \in R_{i-1}^\tau$.

Consider then the sequence $(v_i)_{i=0}^\alpha$ of functions from $\mathscr{A}(X, R_0)$ to the power set of X recurrently defined as follows:

- for every $Y \in \mathscr{A}(X, R_0)$, $v_0(Y) = Y$;
- for every $i \in \{1, \dots, \alpha\}$ and $Y \in \mathscr{A}(X, R_0)$,

$$v_i(Y) = \begin{cases} v_{i-1}(Y) & \text{if } \hat{z}_i \notin v_{i-1}(Y) \\ \emptyset & \text{if } \hat{z}_i \in v_{i-1}(Y) \text{ and } \hat{z}_i \text{ satisfies } \mathfrak{P}_i \\ S(X, R_i; \hat{z}_i) & \text{if } \hat{z}_i \in v_{i-1}(Y) \text{ and } \hat{z}_i \text{ does not satisfy } \mathfrak{P}_i \end{cases} \tag{5}$$

Then, the following facts hold true:

(a) for every $i \in \{0, \dots, \alpha\}$,

- $\{v_i(Y) : Y \in \mathscr{A}(X, R_0)\} \setminus \{\emptyset\} = \mathscr{A}(X, R_i)$,
- for every $Y, Y' \in \mathscr{A}(X, R_0)$ distinct with $v_i(Y) \neq \emptyset$ and $v_i(Y') \neq \emptyset$, we have $v_i(Y) \neq v_i(Y')$;

(b) there exists $Y \in \mathscr{A}(X, R_0)$ such that $v_\alpha(Y) \neq \emptyset$;

(c) $$\bigcap_{i=0}^{\alpha} A(X, R_i) = \bigcup_{Y \in \mathscr{A}(X,R_0), v_\alpha(Y) \neq \emptyset} Y.$$

Note that the assumptions in Theorem 10 imply that $|X| \geq 2$ and that the sequences $(R_i)_{i=0}^{\alpha}$ and $((\hat{y}_i, \hat{z}_i))_{i=1}^{\alpha}$ are made up of distinct elements.

In order to help the reader to get familiar with the content of Theorem 10, it is certainly useful to test its validity in a particular case.

Example 11 Consider $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $(\hat{y}_1, \hat{z}_1) = (6, 4)$, $(\hat{y}_2, \hat{z}_2) = (5, 6)$, $(\hat{y}_3, \hat{z}_3) = (2, 7)$, $(\hat{y}_4, \hat{z}_4) = (8, 2)$, $R_0 = \{(1, 2), (2, 3), (3, 1), (4, 5), (5, 4), (7, 8)\}$ and, for every $i \in \{1, 2, 3, 4\}$, set $R_i = R_{i-1} \cup \{(\hat{y}_i, \hat{z}_i)\}$. Thus, $(R_i)_{i=0}^4$ is a sequence of irreflexive relations on X and $((\hat{y}_i, \hat{z}_i))_{i=1}^4$ is a sequence in X_*^2. Moreover, $(R_i)_{i=0}^4$ and $((\hat{y}_i, \hat{z}_i))_{i=1}^4$ satisfy the assumptions of Theorem 10. A computation shows that

$$\begin{array}{ll}
\mathscr{A}(X, R_0) = \{\{1, 2, 3\}, \{4, 5\}, \{6\}, \{7\}\}, & A(X, R_0) = \{1, 2, 3, 4, 5, 6, 7\}, \\
\mathscr{A}(X, R_1) = \{\{1, 2, 3\}, \{6\}, \{7\}\}, & A(X, R_1) = \{1, 2, 3, 6, 7\}, \\
\mathscr{A}(X, R_2) = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7\}\}, & A(X, R_2) = \{1, 2, 3, 4, 5, 6, 7\}, \\
\mathscr{A}(X, R_3) = \{\{1, 2, 3\}, \{4, 5, 6\}\}, & A(X, R_3) = \{1, 2, 3, 4, 5, 6\}, \\
\mathscr{A}(X, R_4) = \{\{1, 2, 3, 7, 8\}, \{4, 5, 6\}\}, & A(X, R_4) = \{1, 2, 3, 4, 5, 6, 7, 8\}.
\end{array} \tag{6}$$

Thus, we have that

$$\bigcap_{i=0}^{4} A(X, R_i) = \{1, 2, 3, 6\}. \tag{7}$$

Let us built now the sequence $(v_i)_{i=0}^4$ of functions from $\mathscr{A}(X, R_0) = \{\{1, 2, 3\}, \{4, 5\}, \{6\}, \{7\}\}$ to the power set of X defined in the statement of Theorem 10. The next table provides a complete description of the functions v_0, v_1, v_2, v_3 and v_4 by giving their value on each element of $\mathscr{A}(X, R_0)$.

	$\{1, 2, 3\}$	$\{4, 5\}$	$\{6\}$	$\{7\}$
v_0	$\{1, 2, 3\}$	$\{4, 5\}$	$\{6\}$	$\{7\}$
v_1	$\{1, 2, 3\}$	$\varnothing$	$\{6\}$	$\{7\}$
v_2	$\{1, 2, 3\}$	$\varnothing$	$\{4, 5, 6\}$	$\{7\}$
v_3	$\{1, 2, 3\}$	$\varnothing$	$\{4, 5, 6\}$	$\varnothing$
v_4	$\{1, 2, 3, 7, 8\}$	$\varnothing$	$\{4, 5, 6\}$	$\varnothing$

(8)

Of course, we know that, for every $Y \in \mathscr{A}(X, R_0)$, $v_0(Y) = Y$. Let us describe now the way the equalities $v_1(\{6\}) = \{6\}$, $v_2(\{6\}) = \{4, 5, 6\}$ and $v_1(\{4, 5\}) = \varnothing$ are obtained applying (5). All the remaining entries of the table can be deduced with similar arguments. The equality $v_1(\{6\}) = \{6\}$ follows from the fact that $\hat{z}_1 = 4 \notin v_0(\{6\}) = \{6\}$ so that $v_1(\{6\}) = v_0(\{6\}) = \{6\}$. The equality $v_2(\{6\}) = \{4, 5, 6\}$ follows instead from the fact that $\hat{z}_2 = 6 \in v_1(\{6\}) = \{6\}$ and 6 does not satisfy $\mathfrak{P}_2$. Indeed, $6 \in A(X, R_1) = \{1, 2, 3, 6, 7\}$ and $(\hat{z}_2, \hat{y}_2) = (6, 5) \in R_1^{\tau}$. However, $S(X, R_1; 6) = \{6\}$ and, for every $\hat{x}_2 \in A(X, R_1) \setminus S(X, R_1; 6) = \{1, 2, 3, 7\}$, the pair $(\hat{x}_2, \hat{y}_2) = (\hat{x}_2, 5) \notin R_1^{\tau}$. Thus, we have that $v_2(\{6\}) = S(X, R_2; 6) = \{4, 5, 6\}$. Finally, the equality $v_1(\{4, 5\}) = \varnothing$ follows from the fact that $\hat{z}_1 = 4 \in v_0(\{4, 5\}) = \{4, 5\}$ and 4 satisfies $\mathfrak{P}_1$ since $4 \in A(X, R_0)$ and $(\hat{z}_1, \hat{y}_1) = (4, 6) \notin R_0^{\tau}$.

Looking now at (6) and (8), we can observe that the statements (a) and (b) in Theorem 10 are verified. Note also that $v_4(Y) \neq \emptyset$ if and only if $Y = \{1,2,3\}$ or $Y = \{6\}$. As a consequence,

$$\bigcup_{Y \in \mathscr{A}(X,R_0), v_4(Y) \neq \emptyset} Y = \{1,2,3\} \cup \{6\} = \{1,2,3,6\}. \tag{9}$$

By (7) and (9), statement (c) in Theorem 10 is verified, as well.

Let us move on now to the proof of Theorem 10. It is based on several preliminary lemmas. Observe that if R and $\hat{R}$ are irreflexive relations on X with $R \subseteq \hat{R}$, then $R^\tau \subseteq \hat{R}^\tau$, $E_R \subseteq E_{\hat{R}}$ and, for every $x \in X$, $S(X,R;x) \subseteq S(X,\hat{R};x)$. We are going to use those facts along the proofs several times.

Lemma 12 *Let (X,R) be an abstract decision problem and $x \in X$. Then there exists $y^* \in A(X,R)$ such that $(y^*,x) \in R^\tau$.*

Proof Consider the set $Y = \{y \in X : (y,x) \in R^\tau\}$. Since $x \in Y$, we have that $Y \neq \emptyset$. Since R^τ is a transitive and reflexive relation on X and $Y \subseteq X$ is nonempty and finite, there exists $y^* \in Y$ such that, for every $y \in Y$, $(y,y^*) \in R^\tau$ implies $(y^*,y) \in R^\tau$.[7] Of course, since $y^* \in Y$, we have that $(y^*,x) \in R^\tau$. Let us prove then that $y^* \in A(X,R)$. Let $y \in X$ be such that $(y,y^*) \in R^\tau$. We have to prove that $(y^*,y) \in R^\tau$. By $(y^*,x) \in R^\tau$ and transitivity of R^τ, we have that $(y,x) \in R^\tau$ so that $y \in Y$. Thus, $(y^*,y) \in R^\tau$. □

Lemma 13 *Let R and $\hat{R}$ be irreflexive relations on X such that $R \subseteq \hat{R}$ and let $Y \in \mathscr{A}(X,\hat{R})$. Then there exists $\mathscr{B} \subseteq \mathscr{S}(X,R)$ nonempty such that $Y = \bigcup_{Z \in \mathscr{B}} Z$ and $\mathscr{B} \cap \mathscr{A}(X,R) \neq \emptyset$.*

Proof It is known that if E and $\hat{E}$ are equivalence relations on X with $E \subseteq \hat{E}$, then each equivalence class of $\hat{E}$ is the union of equivalence classes of E. As a consequence, since $R \subseteq \hat{R}$ implies $E_R \subseteq E_{\hat{R}}$ and since $Y \in \mathscr{A}(X,\hat{R})$ implies $Y \in \mathscr{S}(X,\hat{R})$, there exists $\mathscr{B} \subseteq \mathscr{S}(X,R)$ nonempty such that $Y = \bigcup_{Z \in \mathscr{B}} Z$. We are left with proving that $\mathscr{B} \cap \mathscr{A}(X,R) \neq \emptyset$. For every $Z \in \mathscr{B}$, let $W(Z) \subseteq X$ be defined as

$$\begin{aligned} W(Z) &= \{x \in X \setminus Z : \forall z \in Z\, (x,z) \in R^\tau\} \\ &= \{x \in X \setminus Z : \exists z \in Z \text{ such that } (x,z) \in R^\tau\}. \end{aligned}$$

Let us prove that, for every $Z \in \mathscr{B}$, $W(Z) \subseteq Y$. Consider $Z \in \mathscr{B}$ and $x \in W(Z)$. Given now $z \in Z$, we have that $(x,z) \in R^\tau$, so that $(x,z) \in \hat{R}^\tau$. Since, in particular, $z \in Y \in \mathscr{A}(X,\hat{R})$, we deduce that $x \in Y$. We conclude then that $W(Z) \subseteq Y$, as desired.

Consider now the relation $\mathfrak{R}$ on $\mathscr{B}$ defined as

$$\mathfrak{R} = \left\{(Z,Z') \in \mathscr{B}^2 : W(Z) \cap Z' \neq \emptyset\right\}.$$

[7] See Lemma 1b in Sen (1970).

We have that $\mathfrak{R}$ is irreflexive. Indeed, for every $Z \in \mathscr{B}$, $W(Z) \cap Z = \emptyset$ so that $(Z, Z) \notin \mathfrak{R}$. Moreover, we also have that $\mathfrak{R}$ is acyclic. Indeed, assume by contradiction that $\mathfrak{R}$ admits a cycle, that is, there exists a sequence $(Z_j)_{j=1}^m$, where $m \geq 3$, $Z_1, \ldots, Z_{m-1}$ are distinct elements of $\mathscr{B}$, $Z_m = Z_1$ and, for every $j \in \{1, \ldots, m-1\}$, $(Z_j, Z_{j+1}) \in \mathfrak{R}$. Then, for every $j \in \{1, \ldots, m-1\}$, $W(Z_j) \cap Z_{j+1} \neq \emptyset$ so that we can pick $x_j \in W(Z_j) \cap Z_{j+1}$. Thus, for every $j \in \{2, \ldots, m-1\}$, $(x_j, x_{j-1}) \in R^\tau$ so that, by transitivity of R^τ, we get $(x_{m-1}, x_1) \in R^\tau$. Moreover, since $x_1 \in W(Z_1) \cap Z_2$ and $x_{m-1} \in Z_m = Z_1$, we also have $(x_1, x_{m-1}) \in R^\tau$. We deduce then $(x_1, x_{m-1}) \in E_R$ that implies the contradiction $Z_2 = Z_1$.

Since $\mathfrak{R}$ is irreflexive and acyclic, there exists $Z^* \in \mathscr{B}$ such that, for every $Z \in \mathscr{B}$, $(Z^*, Z) \notin \mathfrak{R}$.[8] That means that, for every $Z \in \mathscr{B}$, $W(Z^*) \cap Z = \emptyset$. Thus

$$\emptyset = \bigcup_{Z \in \mathscr{B}} (W(Z^*) \cap Z) = W(Z^*) \cap \left(\bigcup_{Z \in \mathscr{B}} Z \right) = W(Z^*) \cap Y.$$

Since $W(Z^*) \subseteq Y$, we conclude that $W(Z^*) = \emptyset$, that is, $Z^* \in \mathscr{A}(X, R)$. □

Lemma 14 *Let R and $\hat{R}$ be irreflexive relations on X. Assume that $R \subseteq \hat{R}$ and that there exists $(\hat{y}, \hat{z}) \in X_*^2$ such that $\hat{R} \setminus R = \{(\hat{y}, \hat{z})\}$. Then, the following facts hold true.*

(*a*) For every $x \in X$, $(x, \hat{y}) \in R^\tau$ if and only if $(x, \hat{y}) \in \hat{R}^\tau$.
(*b*) For every $x \in X$, $(\hat{z}, x) \in R^\tau$ if and only if $(\hat{z}, x) \in \hat{R}^\tau$.
(*c*) For every $Y \in \mathscr{A}(X, R)$, $x \in X \setminus Y$ and $y \in Y$, if $(x, y) \in \hat{R}^\tau$ then $\hat{z} \in Y$.
(*d*) If there exists $\hat{x} \in X$ such that $(\hat{x}, \hat{z}) \in \hat{R}^\tau$ and $(\hat{z}, \hat{x}) \notin R^\tau$, then $S(X, \hat{R}; \hat{z}) \notin \mathscr{A}(X, \hat{R})$.

Proof (*a*) Consider $x \in X$. If $(x, \hat{y}) \in R^\tau$, we have that $(x, \hat{y}) \in \hat{R}^\tau$ since $R^\tau \subseteq \hat{R}^\tau$. Assume now that $(x, \hat{y}) \in \hat{R}^\tau$ and prove that $(x, \hat{y}) \in R^\tau$. If $x = \hat{y}$, then $(x, \hat{y}) \in R^\tau$. Assume then $x \neq \hat{y}$. Thus, there exists a path γ from x to $\hat{y}$ in $\hat{R}$. We show that γ is a path from x to $\hat{y}$ in R so that $(x, \hat{y}) \in R^\tau$. Let $\gamma = (x_i)_{i=1}^m$ where $m \geq 2$, $x_1, \ldots, x_m$ are distinct elements of X with $x_1 = x$ and $x_m = \hat{y}$ and, for every $j \in \{1, \ldots, m-1\}$, $(x_j, x_{j+1}) \in \hat{R}$. Assume by contradiction that γ is not a path in R. Thus, there exists $j^* \in \{1, \ldots, m-1\}$ such that $x_{j^*} = \hat{y}$ and $x_{j^*+1} = \hat{z}$. We obtain then that $x_{j^*} = x_m = \hat{y}$. Since $j^* \neq m$, we have a contradiction.

(*b*) Consider $x \in X$. If $(\hat{z}, x) \in R^\tau$, we have that $(\hat{z}, x) \in \hat{R}^\tau$ since $R^\tau \subseteq \hat{R}^\tau$. Assume now that $(\hat{z}, x) \in \hat{R}^\tau$ and prove that $(\hat{z}, x) \in R^\tau$. If $x = \hat{z}$, then $(\hat{z}, x) \in R^\tau$. Assume then $x \neq \hat{z}$. Thus, there exists a path γ from $\hat{z}$ to x in $\hat{R}$. We show that γ is a path from $\hat{z}$ to x in R so that $(\hat{z}, x) \in R^\tau$. Let $\gamma = (x_i)_{i=1}^m$ where $m \geq 2$, $x_1, \ldots, x_m$ are distinct elements of X with $x_1 = \hat{z}$ and $x_m = x$ and, for every $j \in \{1, \ldots, m-1\}$, $(x_j, x_{j+1}) \in \hat{R}$. Assume by contradiction that γ is not a path in R. Thus, there exists $j^* \in \{1, \ldots, m-1\}$ such that $x_{j^*} = \hat{y}$ and $x_{j^*+1} = \hat{z}$. We obtain then that $x_{j^*+1} = x_1 = \hat{z}$. Since $j^* + 1 \neq 1$, we have a contradiction.

[8] See Proposition 1.4.2 in Bang-Jensen and Gutin (2008).

(c) Consider $Y \in \mathscr{A}(X,R)$, $x \in X \setminus Y$ and $y \in Y$ such that $(x,y) \in \hat{R}^\tau$. Since $x \neq y$, there exists a path γ from x to y in $\hat{R}$. Let $\gamma = (x_i)_{i=1}^m$ where $m \geq 2$, $x_1, \ldots, x_m$ are distinct elements of X with $x_1 = x$ and $x_m = y$ and, for every $j \in \{1, \ldots, m-1\}$, $(x_j, x_{j+1}) \in \hat{R}$. Since $Y \in \mathscr{A}(X,R)$, γ is not a path in (X, R). Thus, there exists $j^* \in \{1, \ldots, m-1\}$ such that $x_{j^*} = \hat{y}$ and $x_{j^*+1} = \hat{z}$. If $j^* + 1 = m$, then $\hat{z} = y \in Y$. If instead $j^* + 1 < m$, the sequence $(x_{j^*+i})_{i=1}^{m-j^*}$ is a path from $\hat{z}$ to y in $(X, \hat{R})$. Thus, $(\hat{z}, y) \in \hat{R}^\tau$. By ($b$) we have that $(\hat{z}, y) \in R^\tau$ and that implies $\hat{z} \in Y$ since $Y \in \mathscr{A}(X,R)$.

(d) Let $\hat{x} \in X$ be such that $(\hat{x}, \hat{z}) \in \hat{R}^\tau$ and $(\hat{z}, \hat{x}) \notin R^\tau$. Assume by contradiction that $S(X, \hat{R}; \hat{z}) \in \mathscr{A}(X, \hat{R})$. Since $(\hat{x}, \hat{z}) \in \hat{R}^\tau$, we have that $\hat{x} \in S(X, \hat{R}; \hat{z})$. Thus, $(\hat{z}, \hat{x}) \in \hat{R}^\tau$ so that, by (b), we get $(\hat{z}, \hat{x}) \in R^\tau$, a contradiction. □

Lemma 15 *Let R and $\hat{R}$ be irreflexive relations on X. Assume that $R \subseteq \hat{R}$ and that there exists $(\hat{y}, \hat{z}) \in X_*^2$ such that $\hat{R} \setminus R = \{(\hat{y}, \hat{z})\}$. Then*

$$\mathscr{A}(X, \hat{R}) \setminus \{S(X, \hat{R}; \hat{z})\} = \mathscr{A}(X, R) \setminus \{S(X, R; \hat{z})\}.$$

Proof Consider $Y \in \mathscr{A}(X,R) \setminus \{S(X,R;\hat{z})\}$ and prove that $Y \in \mathscr{A}(X,\hat{R}) \setminus \{S(X,\hat{R};\hat{z})\}$. Let us prove first that $Y \in \mathscr{S}(X,\hat{R})$. Clearly we have that $Y \neq \emptyset$ so that there exists $y^* \in Y$. Moreover, $\hat{z} \notin Y$. Consider now $x \in X$ such that $(x, y^*) \in E_{\hat{R}}$ and prove that $x \in Y$. Assume by contradiction that $x \notin Y$. From $(x, y^*) \in E_{\hat{R}}$ we have that $(x, y^*) \in \hat{R}^\tau$ so that by Lemma 14(c) we deduce that $\hat{z} \in Y$, a contradiction. Consider now $y \in Y$ and prove that $(y, y^*) \in E_{\hat{R}}$. Since $Y \in \mathscr{A}(X,R)$, we have that $y \in Y$ implies $(y, y^*) \in E_R$. Since $E_R \subseteq E_{\hat{R}}$, we immediately deduce $(y, y^*) \in E_{\hat{R}}$. That finally proves that $Y \in \mathscr{S}(X,\hat{R})$. Since $\hat{z} \notin Y$, we also have that $Y \neq S(X,\hat{R};\hat{z})$. Thus, we are left with proving that $Y \in \mathscr{A}(X,\hat{R})$. If by contradiction $Y \notin \mathscr{A}(X,\hat{R})$, then there exist $x^* \in X \setminus Y$ and $y^* \in Y$ such that $(x^*, y^*) \in \hat{R}^\tau$. By Lemma 14($c$), we have that $\hat{z} \in Y$, a contradiction.

Consider now $Y \in \mathscr{A}(X,\hat{R}) \setminus \{S(X,\hat{R};\hat{z})\}$ and prove that $Y \in \mathscr{A}(X,R) \setminus \{S(X,R;\hat{z})\}$. First of all, note that $\hat{z} \notin Y$. By Lemma 13, there exists $\mathscr{B} \in \mathscr{S}(X,R)$ nonempty such that $Y = \bigcup_{Z \in \mathscr{B}} Z$ and $\mathscr{B} \cap \mathscr{A}(X,R) \neq \emptyset$. Moreover, since $\hat{z} \notin Y$, we have that $S(X,R;\hat{z}) \notin \mathscr{B}$. Let $Z^* \in \mathscr{B} \cap \mathscr{A}(X,R)$. We are going to prove that $|\mathscr{B}| = 1$ so that $Y = Z^* \in \mathscr{A}(X,R) \setminus \{S(X,R;\hat{z})\}$. Assume by contradiction that $|\mathscr{B}| \geq 2$. Then there exists $Z^{**} \in \mathscr{B}$ with $Z^{**} \neq Z^*$. Consider $z^* \in Z^*$ and $z^{**} \in Z^{**}$. Since $Z^* \cap Z^{**} = \emptyset$, $z^{**} \notin Z^*$. Moreover, $z^*, z^{**} \in Y$ so that $(z^{**}, z^*) \in \hat{R}^\tau$. By Lemma 14($c$) applied to $Z^* \in \mathscr{A}(X,R)$ we deduce that $\hat{z} \in Z^*$ so that $\hat{z} \in Y$, a contradiction. □

Lemma 16 *Let R and $\hat{R}$ be irreflexive relations on X. Assume that $R \subseteq \hat{R}$ and that there exists $(\hat{y}, \hat{z}) \in X_*^2$ such that $\hat{R} \setminus R = \{(\hat{y}, \hat{z})\}$. Then the following facts hold true.*

(a) If $\hat{z} \in A(X,R)$ and $(\hat{z}, \hat{y}) \notin R^\tau$, then $S(X,R;\hat{z}) \in \mathscr{A}(X,R)$, $S(X,\hat{R};\hat{z}) \notin \mathscr{A}(X,\hat{R})$ and $\mathscr{A}(X,\hat{R}) = \mathscr{A}(X,R) \setminus \{S(X,R;\hat{z})\}$.

(b) If $\hat{z} \in A(X,R)$, $(\hat{z}, \hat{y}) \in R^\tau$ and there exists $\hat{x} \in A(X,R) \setminus S(X,R;\hat{z})$ such that $(\hat{x}, \hat{y}) \in R^\tau$, then $S(X,R;\hat{z}) \in \mathscr{A}(X,R)$, $S(X,\hat{R};\hat{z}) \notin \mathscr{A}(X,\hat{R})$ and $\mathscr{A}(X,\hat{R}) = \mathscr{A}(X,R) \setminus \{S(X,R;\hat{z})\}$.

(*c*) If $\hat{z} \in A(X,R)$, $(\hat{z},\hat{y}) \in R^\tau$ and, for every $x \in A(X,R) \setminus S(X,R;\hat{z})$, $(x,\hat{y}) \notin R^\tau$, then $S(X,R;\hat{z}) \in \mathscr{A}(X,R)$, $S(X,\hat{R};\hat{z}) \in \mathscr{A}(X,\hat{R})$ and $\mathscr{A}(X,\hat{R}) = (\mathscr{A}(X,R) \setminus \{S(X,R;\hat{z})\}) \cup \{S(X,\hat{R};\hat{z})\}$.

(*d*) If $\hat{z} \notin A(X,R)$, then $S(X,R;\hat{z}) \notin \mathscr{A}(X,R)$, $S(X,\hat{R};\hat{z}) \notin \mathscr{A}(X,\hat{R})$ and $\mathscr{A}(X,\hat{R}) = \mathscr{A}(X,R)$.

Proof (*a*) Assume that $\hat{z} \in A(X,R)$ and $(\hat{z},\hat{y}) \notin R^\tau$. By Theorem 9, $S(X,R;\hat{z}) \in \mathscr{A}(X,R)$. Since $(\hat{y},\hat{z}) \in \hat{R}$, we have that $(\hat{y},\hat{z}) \in \hat{R}^\tau$. Thus, by Lemma 14(*d*), we get $S(X,\hat{R};\hat{z}) \notin \mathscr{A}(X,\hat{R})$. Applying Lemma 15, we finally get $\mathscr{A}(X,\hat{R}) = \mathscr{A}(X,R) \setminus \{S(X,R;\hat{z})\}$.

(*b*) Assume that $\hat{z} \in A(X,R)$, $(\hat{z},\hat{y}) \in R^\tau$ and that there exists $\hat{x} \in A(X,R) \setminus S(X,R;\hat{z})$ such that $(\hat{x},\hat{y}) \in R^\tau$. By Theorem 9, $S(X,R;\hat{z}) \in \mathscr{A}(X,R)$. Since $(\hat{x},\hat{y}) \in R^\tau$ and $(\hat{y},\hat{z}) \in \hat{R}$, we have that $(\hat{x},\hat{y}) \in \hat{R}^\tau$ and $(\hat{y},\hat{z}) \in \hat{R}^\tau$ so that $(\hat{x},\hat{z}) \in \hat{R}^\tau$. Since $\hat{x} \in A(X,R) \setminus S(X,R;\hat{z})$, we have that $(\hat{z},\hat{x}) \notin R^\tau$. By Lemma 14(*d*), we conclude that $S(X,\hat{R};\hat{z}) \notin \mathscr{A}(X,\hat{R})$. Applying Lemma 15, we finally get $\mathscr{A}(X,\hat{R}) = \mathscr{A}(X,R) \setminus \{S(X,R;\hat{z})\}$.

(*c*) Assume that $\hat{z} \in A(X,R)$, $(\hat{z},\hat{y}) \in R^\tau$ and that, for every $x \in A(X,R) \setminus S(X,R;\hat{z})$, $(x,\hat{y}) \notin R^\tau$. By Theorem 9, $S(X,R;\hat{z}) \in \mathscr{A}(X,R)$. Let us prove now that $S(X,\hat{R};\hat{z}) \in \mathscr{A}(X,\hat{R})$. Suppose by contradiction that $S(X,\hat{R};\hat{z}) \notin \mathscr{A}(X,\hat{R})$. Then there exists $\hat{x} \in X$ such that $(\hat{x},\hat{z}) \in \hat{R}^\tau$ and $(\hat{z},\hat{x}) \notin \hat{R}^\tau$. Note that $\hat{x} \notin S(X,R;\hat{z})$ otherwise we would have $(\hat{z},\hat{x}) \in R^\tau$ and then, by Lemma 14(*b*), we get the contradiction $(\hat{z},\hat{x}) \in \hat{R}^\tau$. We can prove now that there exists $\hat{x}^* \in A(X,R) \setminus S(X,R;\hat{z})$ such that $(\hat{x}^*,\hat{z}) \in \hat{R}^\tau$ and $(\hat{z},\hat{x}^*) \notin \hat{R}^\tau$. Indeed, if $\hat{x} \in A(X,R)$, then set $\hat{x}^* = \hat{x}$. If instead $\hat{x} \notin A(X,R)$, by Lemma 12, there exists $\hat{x}^* \in A(X,R)$ such that $(\hat{x}^*,\hat{x}) \in R^\tau$. Thus, $(\hat{x}^*,\hat{x}) \in \hat{R}^\tau$ and by transitivity of $\hat{R}^\tau$, we have $(\hat{x}^*,\hat{z}) \in \hat{R}^\tau$. We also have that $(\hat{z},\hat{x}^*) \notin \hat{R}^\tau$ otherwise, by transitivity of $\hat{R}^\tau$, we get the contradiction $(\hat{z},\hat{x}) \in \hat{R}^\tau$. Finally note that $\hat{x}^* \notin S(X,R;\hat{z})$ otherwise we would have $(\hat{z},\hat{x}^*) \in R^\tau$ and then, by Lemma 14(*b*), we get the contradiction $(\hat{z},\hat{x}^*) \in \hat{R}^\tau$. Now, since $\hat{x}^* \in A(X,R) \setminus S(X,R;\hat{z})$, we know that $(\hat{x}^*,\hat{y}) \notin R^\tau$. However, since $(\hat{z},\hat{y}) \in R^\tau$, we have that $(\hat{z},\hat{y}) \in \hat{R}^\tau$ so that, by transitivity of $\hat{R}^\tau$, $(\hat{x}^*,\hat{y}) \in \hat{R}^\tau$ which implies, by Lemma 14(*a*), the contradiction $(\hat{x}^*,\hat{y}) \in R^\tau$. That completes the proof that $S(X,\hat{R};\hat{z}) \in \mathscr{A}(X,\hat{R})$. Applying Lemma 15, we finally get $\mathscr{A}(X,\hat{R}) = (\mathscr{A}(X,R) \setminus \{S(X,R;\hat{z})\}) \cup \{S(X,\hat{R};\hat{z})\}$.

(*d*) Assume that $\hat{z} \notin A(X,R)$. By Theorem 9, $S(X,R;\hat{z}) \notin \mathscr{A}(X,R)$. Since $\hat{z} \notin A(X,R)$, we know that there exists $\hat{x} \in X$ such that $(\hat{x},\hat{z}) \in R^\tau$ and $(\hat{z},\hat{x}) \notin R^\tau$. Then $(\hat{x},\hat{z}) \in \hat{R}^\tau$ and $(\hat{z},\hat{x}) \notin R^\tau$ and by Lemma 14(*d*) we get $S(X,\hat{R};\hat{z}) \notin \mathscr{A}(X,\hat{R})$. Applying Lemma 15, we finally get $\mathscr{A}(X,\hat{R}) = \mathscr{A}(X,R)$. □

Proof of Theorem 10 We are going to prove the three statements (*a*), (*b*) and (*c*) of Theorem 10. In order to simplify the notation, in what follows we set $\mathscr{A}_0 = \mathscr{A}(X,R_0)$.

(*a*) Let us prove that, for every $i \in \{0,\dots,\alpha\}$, we have that

(α_i) $\{v_i(Y) : Y \in \mathscr{A}_0\} \setminus \{\varnothing\} = \mathscr{A}(X,R_i)$;

(β_i) for every $Y, Y' \in \mathscr{A}_0$ distinct with $v_i(Y) \neq \varnothing$ and $v_i(Y') \neq \varnothing$, we have $v_i(Y) \neq v_i(Y')$.

Let $J = \{i \in \{0, \dots, \alpha\} : (\alpha_i) \text{ and } (\beta_i) \text{ hold true }\}$ and prove that $J = \{0, \dots, \alpha\}$. First of all, note that $0 \in J$. We are then left with proving that if $i \in \{1, \dots, \alpha\}$ is such that $i - 1 \in J$ then $i \in J$. Consider then $i \in \{1, \dots, \alpha\}$ such that $i - 1 \in J$. Thus, (α_{i-1}) and (β_{i-1}) hold true. There are two cases to discuss.

Case 1: Assume first that, for every $Y \in \mathscr{A}_0$, $\hat{z}_i \notin v_{i-1}(Y)$. By (5) we have that, for every $Y \in \mathscr{A}_0$, $v_i(Y) = v_{i-1}(Y)$. By (α_{i-1}) and Theorem 9, we have that

$$\hat{z}_i \notin \bigcup_{Y \in \mathscr{A}_0} v_{i-1}(Y) = \bigcup_{Y \in \mathscr{A}(X, R_{i-1})} Y = A(X, R_{i-1}),$$

so that, applying Lemma 16(*d*), we get $\mathscr{A}(X, R_i) = \mathscr{A}(X, R_{i-1})$. Thus, again using (α_{i-1}), we deduce that

$$\{v_i(Y) : Y \in \mathscr{A}_0\} \setminus \{\varnothing\} = \{v_{i-1}(Y) : Y \in \mathscr{A}_0\} \setminus \{\varnothing\} = \mathscr{A}(X, R_{i-1}) = \mathscr{A}(X, R_i),$$

that proves (α_i). Consider now $Y, Y' \in \mathscr{A}_0$ distinct with $v_i(Y) \neq \varnothing$ and $v_i(Y') \neq \varnothing$. Since $v_{i-1}(Y) = v_i(Y) \neq \varnothing$ and $v_{i-1}(Y') = v_i(Y') \neq \varnothing$, by (β_{i-1}), we know that $v_{i-1}(Y) \neq v_{i-1}(Y')$ so that $v_i(Y) \neq v_i(Y')$. As a consequence, (β_i) is satisfied, as well. We conclude then that $i \in J$, as desired.

Case 2: Assume now that there exists $Y^* \in \mathscr{A}_0$ such that $\hat{z}_i \in v_{i-1}(Y^*)$. Then $v_{i-1}(Y^*) \neq \varnothing$ so that, by (α_{i-1}), $v_{i-1}(Y^*) \in \mathscr{A}(X, R_{i-1})$. Thus,

$$v_{i-1}(Y^*) = S(X, R_{i-1}; \hat{z}_i), \tag{10}$$

and, by Theorem 9, $\hat{z}_i \in A(X, R_{i-1})$. Moreover, for every $Y \in \mathscr{A}_0 \setminus \{Y^*\}$, $\hat{z}_i \notin v_{i-1}(Y)$. Indeed, assume by contradiction that there exists $Y^{**} \in \mathscr{A}_0$ with $Y^{**} \neq Y^*$ such that $\hat{z}_i \in v_{i-1}(Y^{**})$. Using the same argument used for Y^*, we get $v_{i-1}(Y^{**}) = S(X, R_{i-1}; \hat{z}_i)$. Thus, $v_{i-1}(Y^{**}) = v_{i-1}(Y^*)$ and that contradicts (β_{i-1}) since $Y^{**} \neq Y^*$, $v_{i-1}(Y^{**}) \neq \varnothing$ and $v_{i-1}(Y^*) \neq \varnothing$. By (5), we conclude then that

$$\text{for every } Y \in \mathscr{A}_0 \setminus \{Y^*\},\ v_i(Y) = v_{i-1}(Y). \tag{11}$$

In order to complete the proof of Case 2, we need to consider two sub-cases.

Case 2.1: Suppose first that $\hat{z}_i$ satisfies $\mathfrak{P}_i$. By Lemma 16(*a*)(*b*) and (10), we have that

$$\mathscr{A}(X, R_i) = \mathscr{A}(X, R_{i-1}) \setminus \{v_{i-1}(Y^*)\}. \tag{12}$$

By (5), we have that $v_i(Y^*) = \varnothing$. Thus, recalling (α_{i-1}), (11) and (12), we deduce that

$$\begin{aligned} \{v_i(Y) : Y \in \mathscr{A}_0\} \setminus \{\varnothing\} &= \{v_i(Y) : Y \in \mathscr{A}_0 \setminus \{Y^*\}\} \setminus \{\varnothing\} \\ = \{v_{i-1}(Y) : Y \in \mathscr{A}_0 \setminus \{Y^*\}\} \setminus \{\varnothing\} &= \{v_{i-1}(Y) : Y \in \mathscr{A}_0\} \setminus \{\varnothing, v_{i-1}(Y^*)\} \\ = \mathscr{A}(X, R_{i-1}) \setminus \{v_{i-1}(Y^*)\} &= \mathscr{A}(X, R_i), \end{aligned}$$

which proves (α_i).

Consider now $Y, Y' \in \mathscr{A}_0$ distinct with $v_i(Y) \neq \varnothing$ and $v_i(Y') \neq \varnothing$. Since $v_i(Y^*) = \varnothing$, we have that $Y \neq Y^*$ and $Y' \neq Y^*$ so that, by (11), we have that $v_{i-1}(Y) = v_i(Y) \neq \varnothing$ and $v_{i-1}(Y') = v_i(Y') \neq \varnothing$. By (β_{i-1}), we know that

$v_{i-1}(Y) \neq v_{i-1}(Y')$ so that $v_i(Y) \neq v_i(Y')$. As a consequence, (β_i) is satisfied, as well. We conclude then that $i \in J$, as desired.

Case 2.2. Suppose finally that $\hat{z}_i$ does not satisfy $\mathfrak{P}_i$. By Lemma 16(c) and (10), we have that

$$\mathscr{A}(X, R_i) = (\mathscr{A}(X, R_{i-1}) \setminus \{v_{i-1}(Y^*)\}) \cup \{S(X, R_i; \hat{z})\}. \tag{13}$$

By (5), we have that $v_i(Y^*) = S(X, R_i; \hat{z})$. Thus, recalling (α_{i-1}), (11) and (13), we deduce that

$$\begin{aligned}
\{v_i(Y) : Y \in \mathscr{A}_0\} \setminus \{\varnothing\} &= \Big(\{v_i(Y) : Y \in \mathscr{A}_0 \setminus \{Y^*\}\} \cup \{S(X, R_i; \hat{z})\}\Big) \setminus \{\varnothing\} \\
&= \Big(\{v_i(Y) : Y \in \mathscr{A}_0 \setminus \{Y^*\}\} \setminus \{\varnothing\}\Big) \cup \{S(X, R_i; \hat{z})\} \\
&= \Big(\{v_{i-1}(Y) : Y \in \mathscr{A}_0 \setminus \{Y^*\}\} \setminus \{\varnothing\}\Big) \cup \{S(X, R_i; \hat{z})\} \\
&= \Big(\{v_{i-1}(Y) : Y \in \mathscr{A}_0\} \setminus \{\varnothing, v_{i-1}(Y^*)\}\Big) \cup \{S(X, R_i; \hat{z})\} \\
&= (\mathscr{A}(X, R_{i-1}) \setminus \{v_{i-1}(Y^*)\}) \cup \{S(X, R_i; \hat{z})\} = \mathscr{A}(X, R_i),
\end{aligned}$$

which proves (α_i).

Consider now $Y, Y' \in \mathscr{A}_0$ distinct with $v_i(Y) \neq \varnothing$ and $v_i(Y') \neq \varnothing$. If $Y \neq Y^*$ and $Y' \neq Y^*$, then, by (11), $v_{i-1}(Y) = v_i(Y) \neq \varnothing$ and $v_{i-1}(Y') = v_i(Y') \neq \varnothing$. By (β_{i-1}), we know that $v_{i-1}(Y) \neq v_{i-1}(Y')$ so that $v_i(Y) \neq v_i(Y')$. Assume now that one between Y and Y' equals Y^*. Without loss of generality, let $Y' = Y^*$. We know that $v_{i-1}(Y^*) \neq \varnothing$ and, using again (11), $v_{i-1}(Y) = v_i(Y) \neq \varnothing$. Thus, by (β_{i-1}), $v_{i-1}(Y) \neq v_{i-1}(Y^*)$. By (α_{i-1}), we have that $v_{i-1}(Y), v_{i-1}(Y^*) \in \mathscr{S}(X, R_{i-1})$ so that $v_{i-1}(Y) \cap v_{i-1}(Y^*) = \varnothing$. In particular, from $\hat{z}_i \in v_{i-1}(Y^*)$ we get $\hat{z}_i \notin v_{i-1}(Y)$. Since $v_i(Y^*) = S(X, R_i; \hat{z})$ and $v_i(Y) = v_{i-1}(Y)$, we have that $\hat{z}_i \in v_i(Y^*)$ and $\hat{z}_i \notin v_i(Y)$ so that $v_i(Y^*) \neq v_i(Y)$. As a consequence, (β_i) is satisfied, as well. We conclude then that $i \in J$, as desired.

(b) Let us prove that there exists $Y^* \in \mathscr{A}_0$ such that $v_\alpha(Y^*) \neq \varnothing$. By Theorem 9, we know that $\mathscr{A}(X, R_\alpha) \neq \varnothing$. Thus, by (a), we get $\{v_\alpha(Y) : Y \in \mathscr{A}_0\} \setminus \{\varnothing\} \neq \varnothing$. That immediately implies the existence of $Y^* \in \mathscr{A}_0$ such that $v_\alpha(Y^*) \neq \varnothing$.

(c) Let us prove the equality $\bigcap_{i=0}^{\alpha} A(X, R_i) = \bigcup_{Y \in \mathscr{A}_0, v_\alpha(Y) \neq \varnothing} Y$. Note that (b) implies that $\{Y \in \mathscr{A}_0, v_\alpha(Y) \neq \varnothing\} \neq \varnothing$. Some preliminary claims are needed.

Claim 1. Let $Y \in \mathscr{A}_0$ and $i^* \in \{0, \dots, \alpha\}$. If $v_{i^*}(Y) \neq \varnothing$, then, for every $i \in \{0, \dots, i^*\}$, $v_i(Y) \neq \varnothing$.

Proof Assume that $v_{i^*}(Y) \neq \varnothing$. If $i^* = 0$, there is nothing to prove so that we can suppose $i^* > 0$. Consider $J = \{i \in \{0, \dots, \alpha\} : v_i(Y) \neq \varnothing\}$ and prove that $\{0, \dots, i^*\} \subseteq J$. Surely $i^* \in J$. We are left with proving that, for every $i \in \{1, \dots, i^*\}$, if $i \in J$, then $i - 1 \in J$. Consider then $i \in J$ so that $v_i(Y) \neq \varnothing$. If by contradiction $i - 1 \notin J$, then $v_{i-1}(Y) = \varnothing$. Thus, $\hat{z}_i \notin v_{i-1}(Y)$ so that, by (5), we get $v_i(Y) = v_{i-1}(Y) = \varnothing$, a contradiction. □

Claim 2. Let $Y \in \mathscr{A}_0$. Then, for every $i \in \{1, \ldots, \alpha\}$, $v_i(Y) \neq \emptyset$ implies $v_{i-1}(Y) \subseteq v_i(Y)$.

Proof Let $i \in \{1, \ldots, \alpha\}$, assume that $v_i(Y) \neq \emptyset$ and prove that $v_{i-1}(Y) \subseteq v_i(Y)$. If $\hat{z}_i \notin v_{i-1}(Y)$, then, by (5), we have that $v_i(Y) = v_{i-1}(Y)$, so that $v_{i-1}(Y) \subseteq v_i(Y)$. If $\hat{z}_i \in v_{i-1}(Y)$ and $\hat{z}_i$ does not satisfy $\mathfrak{P}_i$, then, by (5), we have that $v_i(Y) = S(X, R_i; \hat{z})$. By (a) we know that $v_{i-1}(Y) \in \mathscr{A}(X, R_{i-1})$ so that $v_{i-1}(Y) = S(X, R_{i-1}; \hat{z})$. Since $S(X, R_{i-1}; \hat{z}) \subseteq S(X, R_i; \hat{z})$, we conclude that $v_{i-1}(Y) \subseteq v_i(Y)$. We finally observe that the situation where $\hat{z}_i \in v_{i-1}(Y)$ and $\hat{z}_i$ satisfies $\mathfrak{P}_i$ cannot occur. Indeed, assume by contradiction that $\hat{z}_i \in v_{i-1}(Y)$ and $\hat{z}_i$ satisfies $\mathfrak{P}_i$. Then, by (5), we get the contradiction $v_i(Y) = \emptyset$. □

Claim 3. Let $Y \in \mathscr{A}_0$. If $v_\alpha(Y) \neq \emptyset$, then $\bigcap_{i=0}^{\alpha} v_i(Y) = Y$.

Proof Assume that $v_\alpha(Y) \neq \emptyset$. By Claim 1 we know that, for every $i \in \{0, \ldots, \alpha\}$, $v_i(Y) \neq \emptyset$. By Claim 2 we deduce then that, for every $i \in \{1, \ldots, \alpha\}$, $v_{i-1}(Y) \subseteq v_i(Y)$. As a consequence, $\bigcap_{i=0}^{\alpha} v_i(Y) = v_0(Y) = Y$. □

Once all the previous claims are established, we have all the tools to complete the proof of (c). First of all, applying Theorem 9, (a), Claim 3 and some basic facts from set theory, we have that

$$\bigcap_{i=0}^{\alpha} A(X, R_i) = \bigcap_{i=0}^{\alpha} \left(\bigcup_{Y \in \mathscr{A}(X, R_i)} Y \right) = \bigcap_{i=0}^{\alpha} \left(\bigcup_{Y \in \mathscr{A}_0} v_i(Y) \right) \supseteq \bigcup_{Y \in \mathscr{A}_0} \left(\bigcap_{i=0}^{\alpha} v_i(Y) \right)$$
$$= \bigcup_{Y \in \mathscr{A}_0, v_\alpha(Y) \neq \emptyset} \left(\bigcap_{i=0}^{\alpha} v_i(Y) \right) = \bigcup_{Y \in \mathscr{A}_0, v_\alpha(Y) \neq \emptyset} Y.$$

That proves then the inclusion $\bigcup_{Y \in \mathscr{A}_0, v_\alpha(Y) \neq \emptyset} Y \subseteq \bigcap_{i=0}^{\alpha} A(X, R_i)$. Thus, we are left with proving that $\bigcap_{i=0}^{\alpha} A(X, R_i) \subseteq \bigcup_{Y \in \mathscr{A}_0, v_\alpha(Y) \neq \emptyset} Y$. Assume by contradiction that there exists $y^* \in \bigcap_{i=0}^{\alpha} A(X, R_i)$ such that $y^* \notin \bigcup_{Y \in \mathscr{A}_0, v_\alpha(Y) \neq \emptyset} Y$. Thus, for every $i \in \{0, \ldots, \alpha\}$, $y^* \in A(X, R_i)$. In particular, $y^* \in A(X, R_0)$ so that there exists $Y^* \in \mathscr{A}_0$ with $v_\alpha(Y^*) = \emptyset$ such that $y^* \in Y^*$. Since $v_0(Y^*) = Y^* \neq \emptyset$ and $v_\alpha(Y^*) = \emptyset$, there exists $i^* \in \{1, \ldots, \alpha\}$ such that $v_{i^*-1}(Y^*) \neq \emptyset$ and $v_{i^*}(Y^*) = \emptyset$. By Claim 1, for every $i \in \{0, \ldots, i^* - 1\}$, $v_i(Y^*) \neq \emptyset$. Thus, by Claim 2, for every $i \in \{1, \ldots, i^* - 1\}$, $v_{i-1}(Y^*) \subseteq v_i(Y^*)$. Since $y^* \in Y^* = v_0(Y^*)$, we deduce then that, for every $i \in \{0, \ldots, i^* - 1\}$, $y^* \in v_i(Y^*)$. In particular, $y^* \in v_{i^*-1}(Y^*)$. Since $v_{i^*-1}(Y^*) \neq \emptyset$ and $v_{i^*}(Y^*) = \emptyset$, by (5), we deduce that $\hat{z}_{i^*} \in v_{i^*-1}(Y^*)$ and $\hat{z}_{i^*}$ satisfies $\mathfrak{P}_{i^*}$. By Lemma 16$(a)(b)$, we get $S(X, R_{i^*}; \hat{z}_{i^*}) \notin \mathscr{A}(X, R_{i^*})$ so that, by Theorem 9, $S(X, R_{i^*}; \hat{z}_{i^*}) \cap A(X, R_{i^*}) = \emptyset$. From $y^* \in v_{i^*-1}(Y^*)$, $\hat{z}_{i^*} \in v_{i^*-1}(Y^*)$ and (a), we deduce that $S(X, R_{i^*-1}; y^*) = S(X, R_{i^*-1}; \hat{z}_{i^*})$. Since $S(X, R_{i^*-1}; \hat{z}_{i^*}) \subseteq S(X, R_{i^*}; \hat{z}_{i^*})$, we finally have that $S(X, R_{i^*-1}; y^*) \cap A(X, R_{i^*}) = \emptyset$. Thus, $y^* \notin A(X, R_{i^*})$, a contradiction. □

The next result provides a first interesting consequence of Theorem 10.

Theorem 17 *Let X be a nonempty and finite set, $\alpha \in \mathbb{N}$ and $(R_i)_{i=0}^{\alpha}$ be a sequence of irreflexive relations on X. Assume that $\mathscr{A}(X, R_\alpha) = \{X\}$ and, for every $i \in \{1, \dots, \alpha\}$, $R_{i-1} \subseteq R_i$ and $|R_i \setminus R_{i-1}| = 1$. Then, there exists $Y^* \in \mathscr{A}(X, R_0)$, such that $\bigcap_{i=0}^{\alpha} A(X, R_i) = Y^*$.*

Proof For every $i \in \{1, \dots, \alpha\}$, let us denote by $(\hat{y}_i, \hat{z}_i) \in X_*^2$ the unique element of $R_i \setminus R_{i-1}$. The sequences $(R_i)_{i=0}^{\alpha}$ and $((\hat{y}_i, \hat{z}_i))_{i=1}^{\alpha}$ satisfy then the assumptions of Theorem 10. We can consider then the sequence of functions $(v_i)_{i=0}^{\alpha}$ as defined in the statement of Theorem 10. By Theorem 10(a), $\{v_\alpha(Y) : Y \in \mathscr{A}(X, R_0)\} \setminus \{\varnothing\} = \mathscr{A}(X, R_\alpha) = \{X\}$ and, for every $Y, Y' \in \mathscr{A}(X, R_0)$, $v_\alpha(Y) = v_\alpha(Y') \neq \varnothing$ implies $Y = Y'$. As a consequence, we have that there exists a unique $Y^* \in \mathscr{A}(X, R_0)$ such that $v_\alpha(Y^*) \neq \varnothing$. Thus, by Theorem 10($c$), we finally get $\bigcap_{i=0}^{\alpha} A(X, R_i) = Y^*$. □

Using Theorem 17, we can finally deduce Theorem 2.

Proof of Theorem 2 Since $R_0 = \varnothing$, we have that $\mathscr{A}(X, R_0) = \{\{x\} : x \in X\}$. Since $R_\alpha = X_*^2$, we have that $\mathscr{A}(X, R_\alpha) = \{X\}$. Thus, by Theorem 17, there exists $x^* \in X$ such that $\bigcap_{i=0}^{\alpha} A(X, R_i) = \{x^*\}$. □

8 Further generalizations of Theorems 2 and 3

Even though the admissible set is defined for abstract decision problems, it is clear that the set

$$A(X, R) = \{x \in X : \forall y \in X,\ (y, x) \in R^\tau \text{ implies } (x, y) \in R^\tau\}$$

can also be considered for any pair (X, R), where X is a nonempty and finite set and R is a relation (not necessarily irreflexive) on X. The next results provide simple generalizations of Theorems 2 and 3. Similar generalizations hold true for Theorems 10 and 17, as well.

Theorem 18 *Let X be a nonempty and finite set, $\alpha \in \mathbb{N}$ and $(R_i)_{i=0}^{\alpha}$ be a sequence of relations on X. Assume that $R_0 \setminus X_d^2 = \varnothing$, $R_\alpha \setminus X_d^2 = X_*^2$, and, for every $i \in \{1, \dots, \alpha\}$, $R_{i-1} \setminus X_d^2 \subseteq R_i \setminus X_d^2$ and $|(R_i \setminus X_d^2) \setminus (R_{i-1} \setminus X_d^2)| \leq 1$. Then, there exists $x^* \in X$, such that $\bigcap_{i=0}^{\alpha} A(X, R_i) = \{x^*\}$.*

Proof For every $i \in \{0, \dots, \alpha\}$, let $\hat{R}_i = R_i \setminus X_d^2$ and note that $\hat{R}_i^\tau = R_i^\tau$ so that $A(X, \hat{R}_i) = A(X, R_i)$. The sequence $(\hat{R}_i)_{i=0}^{\alpha}$ is a sequence of irreflexive relations on X such that $\hat{R}_0 = \varnothing$, $\hat{R}_\alpha = X_*^2$, and, for every $i \in \{1, \dots, \alpha\}$, $\hat{R}_{i-1} \subseteq \hat{R}_i$ and $|\hat{R}_i \setminus \hat{R}_{i-1}| \leq 1$.

Assume first that $|X| = 1$. Then there exists $x^* \in X$ such that $X = \{x^*\}$. Moreover, for every $i \in \{0, \dots, \alpha\}$, $\hat{R}_i = \varnothing$ so that $A(X, R_i) = A(X, \hat{R}_i) = X = \{x^*\}$. We conclude then that $\bigcap_{i=0}^{\alpha} A(X, R_i) = \{x^*\}$.

Assume now that $|X| \geq 2$. It is simple to show that there exists a sequence $(S_j)_{j=0}^{\beta}$ of irreflexive relations on X having the following properties: $\beta = |X_*^2|$; $S_0 = \emptyset$; $S_\beta = X_*^2$; for every $j \in \{1, \ldots, \beta\}$, $S_{j-1} \subseteq S_j$ and $|S_j \setminus S_{j-1}| = 1$; $\{S_j : j \in \{0, \ldots, \beta\}\} = \{\hat{R}_i : i \in \{0, \ldots, \alpha\}\}$. By Theorem 2 applied to $(S_j)_{j=0}^{\beta}$, we deduce that there exists $x^* \in X$, such that $\bigcap_{j=0}^{\beta} A(X, S_j) = \{x^*\}$. Since $\bigcap_{j=0}^{\beta} A(X, S_j) = \bigcap_{i=0}^{\alpha} A(X, \hat{R}_i) = \bigcap_{i=0}^{\alpha} A(X, R_i)$, we conclude that $\bigcap_{i=0}^{\alpha} A(X, R_i) = \{x^*\}$. □

Theorem 19 *Let X be a nonempty and finite set, $\alpha \in \mathbb{N}$ and $(R_i)_{i=0}^{\alpha}$ be a sequence of relations on X. Assume that, for every $i \in \{1, \ldots, \alpha\}$, $R_{i-1} \setminus X_d^2 \subseteq R_i \setminus X_d^2$. Then, $\bigcap_{i=0}^{\alpha} A(X, R_i) \neq \emptyset$.*

Proof For every $i \in \{0, \ldots, \alpha\}$, let $\hat{R}_i = R_i \setminus X_d^2$ and note that $\hat{R}_i^\tau = R_i^\tau$ so that $A(X, \hat{R}_i) = A(X, R_i)$. By assumption, for every $i \in \{1, \ldots, \alpha\}$, we have that $\hat{R}_{i-1} \subseteq \hat{R}_i$. By Theorem 3, we deduce that $\bigcap_{i=0}^{\alpha} A(X, \hat{R}_i) \neq \emptyset$. Thus, we conclude that $\bigcap_{i=0}^{\alpha} A(X, R_i) \neq \emptyset$. □

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